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DIFFUSION IN THE BOUNDARY LAYER ON A PLATE WITH INHOMOGENEOUS CHEMICAL PROPERTIES*

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Stationary convective diffusion of a substance dissolved in a viscous incompressible fluid in laminar flow over a flat plate is considered. An irreversible chemical transformation of the substance takes place at the plate face. The reaction kinetics are nonlinear and its constant rate depends on temperature. The reaction thermal effect is taken into account. The Reynolds, Péclet, and the thermal Prandtl numbers are assumed large, thus ensuring the presence of respective boundary layers on the plate. The problem is reduced to a nonlinear integral equation. An analog of similar flow in a round pipe is formulated. Solution of these equations is derived by the method of iteration. Exact lower and upper estimates of the problem solution are obtained. Conditions under which the concentration of substance on the surface diminishes as the distance from the plate leading edge increases are obtained. Numerical calculation results are adduced.

Neglecting the heat transfer in the plate, we obtain the known (e.g., /1/) equations and boundary conditions

$$D\frac{\partial^2 c}{\partial y^2}=u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y} \tag{1}$$

$$y\to\infty,\quad c\to C_\infty;\quad x=0,\quad y\neq 0,\quad c=C_\infty$$

$$y=0,\quad D\,(\partial c/\partial y)=kF(c),\quad k=k(T)$$

$$a\frac{\partial^2 T}{\partial y^2}=u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} \tag{2}$$

$$y\to\infty,\quad T\to T_\infty;\quad x=0,\ y\neq 0,\quad T=T_\infty$$

$$y=0,\quad -\lambda\,(\partial T/\partial y)=QD\,(\partial c/\partial y)$$

where x and y are Cartesian coordinates along and across the plate, u and v velocity components of fluid along axes x and y, respectively, c is the concentration of dissolved substance, T is the fluid temperature, D, a, λ are, the diffusion, thermal diffusivity, and heat conductivity coefficients, respectively, $kF(c)$ is the reaction rate, k is the constant of reaction rate dependent on temperature, and Q is the reaction thermal effect.

We assume the coefficient of the fluid kinematic viscosity ν to be independent of temperature, and the hydrodynamic problem of determining u and v to be separate from the diffusion and thermal problems. Formulas (1) and (2) are closed by specifying u and v. Systems (1) and (2) are related by the boundary conditions at the plate surface. We shall seek the solution in the class of continuous functions.

Let us, first, consider the case when the plate surface is isothermic (in terms of formula (2) this condition is satisfied when $Q=0$, although in practice it can be achieved also when $Q\neq 0$ by feeding or extracting heat), and value of k is a specified function of x

$$y=0,\quad T=\text{const},\ k=\varkappa(x) \tag{3}$$

Actually, the dependence of k on x is not a priori known and is determined in the course of solution in the form $k=k[T(x)]$. However, some of the problems reduce, as shown below, to problem (1), (3), whether their formulation is approximate, or sometimes, an exact one.

Everywhere below we assume that $\mathrm{Pr}^{1/3}\gg 1$ ($\mathrm{Pr}=\nu/D$ is the Prandtl number), i.e. that the diffusion in boundary layer is considerably thinner than the hydrodynamic. Problem (1),(3) was solved analytically, when it was possible to restrict the series in y for u and v to the first terms /2, 3/, and when $F(c)=c$ for $\varkappa(x)=\mathbf{const}$, while for $F(c)=c^n$ ($n=2$ and 0.5), numerically /4/. The analog of problem (1),(3) for smooth particles when $\varkappa(x)=\text{const}$, usually at large Péclet numbers, was investigated in a number of publications (e.g., /5/). Also known are solutions in special cases of

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$\varkappa(x) \not\equiv \text{const}$ (in /2/ for $\varkappa = 0$ in the band adjoining the plate leading edge, and downstream of that band when $\varkappa = \infty$, and in /6/ for staggered bands).

Taking for u and v asymptotic expressions as $y \to 0$ in the Balsius problem /2/, from (1), (3), proceeding as in /4/, we obtain the integral equation

$$C = \mathbf{A}[C], \quad \mathbf{A}[C] \equiv \mathbf{A}[C, R, F] = C_\infty - b\int_0^t \frac{R(s) F[C(s)]\, ds}{(t-s)^{2/3}} \tag{4}$$

$$C \equiv c[x(t), y]_{y=0}, \quad t = 4(2\alpha)^{1/2} D(u_0 x)^{3/2}/(3v^{1/2}), \quad \alpha = 0.332$$
$$R(t) = t^{1/3}\varkappa[x(t)], \quad b = v^{1/3}[2\alpha^{1/3}\Gamma(2/3) u_0 D^{2/3}]^{-1}$$

where $(\Gamma(x)$ is the gamma function and U_0 is the oncoming stress velocity.

For the flow in a tube of radius r_0 in a similar formulation we again obtain Eq.(4) in which

$$t = 2D(u_0/r_0)^{1/2}x, \quad b = r_0^{1/3}[(12)^{1/3}\Gamma(2/3)u_0^{1/3}D]^{-1}, \quad R(t) = \varkappa[x(t)] \tag{5}$$

Assuming that functions $\varkappa[x(t)]$, $t^{2/3}dR/dt$ are continuous for $t \geqslant 0$, and dF/dc for $0 < c \leqslant C_\infty$, with conditions $F(0) = 0$, $dF/dc \geqslant 0$, $0 < c \leqslant C_\infty$ and constraint $dR/dt \geqslant 0, t > 0$, it is possible to prove the inequality

$$dC/dt \leqslant 0, \quad t > 0 \tag{6}$$

Using Eq.(4) and inequality (6), we obtain the exact estimates

$$C_0(t) \leqslant C(t) \leqslant C_1(t) \tag{7}$$
$$\left(C_0 = C_\infty - \frac{2\pi}{\sqrt{3}} bR(t) t^{1/3} F(C_0), \quad C_1 = C_\infty - bF(C_1)\int_0^t \frac{R(s)\,ds}{(t-s)^{2/3}}\right)$$

We derive the solution of Eq.(4) by the method of iteration /7/, taking C_0, C_1 as the input estimates

$$C_{2k} = \max\{C_{2(k-1)}, \mathbf{A}[C_{2k-1}]\}, \quad C_{2k+1} = \min\{C_{2k-1}, \mathbf{A}[C_{2k}]\}, \quad k = 1, 2, \ldots \tag{8}$$

Proceeding as in /7/, we can prove the inequalities

$$C_{2(i-1)} \leqslant C_{2i}, \quad C_{2i-1} \geqslant C_{2i+1}, \quad i = 1, 2, \ldots; \quad C_{2j} \leqslant C \leqslant C_{2j+1}, \quad j = 0, 1, \ldots$$

and the convergence of C_n to C as $n \to \infty$. Any continuous functions that satisfy the condition $0 < C_0 \leqslant C \leqslant C_1 \leqslant C_\infty$ can be taken for C_0 and C_1. For instance, we can use the inequality

$$j/j_\infty \equiv P(C_r) \leqslant 1, \quad C_r \equiv C/C_\infty$$
$$P(C_r) \equiv P(z, C_r) = \frac{2\pi}{\sqrt{3}} z^{2/3} q(z)\Phi(C_r), \quad j \equiv D\left(\frac{\partial c}{\partial y}\right)_{y=0}$$
$$j_\infty = DC_\infty\left(\frac{3}{2}\right)^{2/3}(\alpha\,\mathrm{Pr})^{1/3}\left(\frac{u_0}{vx}\right)^{1/2}\Gamma^{-1}(1/3)$$

where j_∞ is a local diffusion stream flowing on the plate under condition of total absorption /2/. The dimensionless functions q and Φ, and the variable z are defined by conditions

$$\varkappa[x(t)]F(c) = \varkappa_0 C_\infty q(z)\Phi(C_r), \quad z = (b\varkappa_0)^{3/2} t$$

We, then, obtain the more precise upper bound C_1' of the solution of Eq.(4)

$$C \leqslant C_1' = \begin{cases} C_1, & P(1) \leqslant 1 \\ \min(C_1, C_*), & P(1) > 1 \end{cases}$$

where C_* is defined in the last case by the equation $P(C_*/C_\infty) = 1$. Estimate C_1' becomes meaningful, i.e. lower than C_1 only for fairly large z.

It is expedient to take at the n-th step $C_n^* = (C_n + C_{n-1})/2$, as the approximate solution, since then the relative error is defined by the quantity $\delta_n = |C_n - C_{n-1}|/(2\min(C_n, C_{n-1}))$.

The estimate $C_0(t)$ for $\varkappa(t) \equiv \text{const}$ is the same as the approximate solution of Eq.(4) obtained in /4/ by the method of uniformly accessible surface.

Let us consider, as an example, the case in which variability of the constant is due to its dependence on temperature in conformity with the Arrenhius law

$$k = k_0 \exp[-E/(R_0 T)], \quad k_0 = \text{const} \tag{9}$$

Let us now consider the case of viscous fluids for which $\mathrm{Pr}_T^{1/3} \gg 1, \mathrm{Pr}_T = v/a$, when it is possible to derive from Eqs.(1) and (2) the relation between c and T at the surface

$$y = 0, \quad T = T_\infty\left(1 + \varepsilon - \varepsilon \frac{c}{C_\infty}\right), \quad \varepsilon = \frac{QDC_\infty}{T_\infty\lambda}\left(\frac{D}{a}\right)^{-1/2} \tag{10}$$

If the reaction is endothermic, i.e. $Q < 0$, then the subsitution of (9) into Eq.(4) with allowance for relation (10) yields

$$C = \mathbf{A}\,[C, R_1, \Psi] \tag{11}$$
$$\Psi(C) = F(C)\exp\left[g\varepsilon\frac{1 - C/C_\infty}{1 + \varepsilon(1 - C/C_\infty)}\right], \quad g = \frac{E}{R_0T_\infty}, \quad R_1(t) = k_0e^{-g}t^{-1/2}$$

with condition

$$d\psi/dC \geqslant 0, \; 0 < C \leqslant C_\infty \tag{12}$$

satisfied.

When the reaction is exothermic, i.e. $Q > 0$, we first find the approximate expression for $T(t)$, taking into account that under usual conditions $\varepsilon \ll 1$ /4/. In that case

$$T(t) \approx T_1 = T_\infty(1 + \varepsilon - \varepsilon C^\circ/C_\infty) \tag{13}$$

$$C^\circ = \mathbf{A}\,[C^\circ, R_1, F] \tag{14}$$

Substituting (8) into Eq.(4) with allowance for (13), for the approximate determination we have

$$C = A\,[C, R_2, F], \; R_2(t) = R_1(t)\exp[g(1 - T_\infty/T_1)] \tag{15}$$

Since $dR_1/dt \geqslant 0$, hence from inequality (6) it follows in conformity to Eq.(14) that $dC^\circ/dt \leqslant 0$, i.e. with allowance for (13) we have $dT_1/dt \geqslant 0$. Consequently

$$dR_2/dt \geqslant 0 \tag{16}$$

Thus problem (1), (2), (8) reduces in the case of exact formulation with $\varepsilon < 0$ to problem (1), (3) in the particular case of $\varkappa(x) \equiv \mathrm{const}$ with the substitution of $\psi(C)$ for $F(C)$, and in the case of approximate formulation with $\varepsilon > 0$, but $\varepsilon \ll 1$, and with $\varkappa(x) = k_0e^{-g}\exp[g(1 - T_\infty/T_1)]$ it reduces to the same problem. By virtue of conditions (12) and (16) and of inequalities $dR_1/dt \geqslant 0$, $dF/dC \geqslant 0$ it is possible to apply to Eqs.(11), (15), and (14) the respective estimates (7) and method (8).

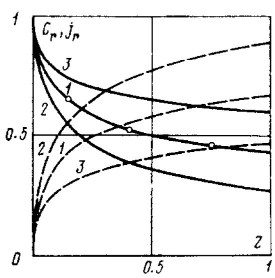

Fig.1

The dependence of $C_r = C/C_\infty$ and $j_r = j/j_\infty$ on z is shown in Fig. 1 by solid and dash lines, respectively in the case of $F(C) = C^2$ and $g = 50$. The error of computation with an accuracy to C_2^* and $z \leqslant 1$ did not exceed 0.06 for curve 1 ($\varepsilon = 0, C_r - z$), and 0.1 for curves 2 and 3 ($\varepsilon = 0.05$ and $-0.05, C_r - z$). The small circles correspond to numerical solution of Eq.(4) obtained in /4/ with $\varepsilon = 0$. According to these data the discrepancy in the case of curve 1 ($C_r - z$) does not exceed 0.005 ($z \leqslant 1$).

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